Lecture Notes
New Keynesian Economics

David McCarthy*

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*These lecture notes are based on Edouard Challe's lectures for Macroeconomics II – New Keynesian Economics at the EUI. Please report typos, mistakes, and comments to david.mccarthy@eui.eu.

Contents

1 Introduction

This course introduces students to the New Keynesian model. We will derive the New Keynesian Phillips Curve from nominal rigidities, and then study how it interacts with aggregate demand to jointly determine output, employment and inflation over the business cycle. In so doing we will examine various dimensions of monetary and fiscal policies, both in normal times and during liquidity traps.

Disclaimer: These notes follow closely the Macro II - New Keynesian Economics lectures by Edouard Challe. They should be seen as a complement to the lecture, not as a substitute, and I hope they will help you in studying the lecture's topics. The main part of the notes covers the material discussed in class. The appendix provides an overview of linearization techniques that students will be required to use in problem sets. If there are any mistakes or typos in the notes or conflicts with information provided in the lecture, please drop me an email david.mccarthy@eui.eu. When in doubt (or for what is relevant in the exam), always rely on the lecture material provided by Edouard.

These notes are loosely built around a number of references. The main textbook for the material covered here is Galí (2015), Challe (2019) is an useful textbook for an advanced undergraduate-level presentation of the New Keynesian Model. In addition, we cite relevant research papers where applicable.

2 The Basic New Keynesian Model

The model consists of households, firms, and a central bank. Households supply labor, purchase goods for consumption, and hold bonds, while firms hire labor and produce and sell differentiated products in monopolistic competitive goods markets. The basic model of monopolistic competition is drawn from Dixit and Stiglitz (1977). The model of price stickiness is taken from Calvo (1983). Each firm sets the price of the good it produces, but not all firms reset their price in each period. Households and firms behave optimally. Households maximize the expected present value of utility, and firms maximize profits. There is also a central bank that controls the nominal interest rate.

2.1 Households

Taking as given prices of all consumption goods $P_t(i), i \in [0,1]$, bond price, ${Q_t}^1$, nominal wage rate, W_t, the representative household solves the following optimization problem:

$$\max_{\{C_t, N_t, B_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t\right)$$

where

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}, \quad \forall t, \quad \varepsilon > 1 \tag{1}$$

s.t.

$$\int_0^1 P_t(i) C_t(i) di + Q_t B_t \leq B_{t-1} + W_t N_t + D_t, \quad \forall t \tag{2}$$

$$B_{-1} \text{ given,}$$

where N_t are total hours worked and B_t is the quantity of bonds bought. Equation (1) is called Dixit-Stiglitz or CES (Constant Elasticity of Substitution) aggregator.[2] For simplicity, we can think about this as a firm that produces a final good, C_t, using as intermediates all the varieties $C_t(i)$, $i \in [0,1]$.

To have a well-defined solution, the above sequence of budget constraints is supplemented

[1] $Q_t = \frac{1}{1+i_t}$ is the price paid in time t for a bond that gives safe return of 1 at time t+1.

[2] The intuition behind this operator is very close to a CES utility function or production function.

with the following No-Ponzi-Game condition:

$$\lim_{k\to\infty} \mathbb{E}_t \left[\Lambda_{t,t+k}\frac{B_{t+k}}{P_{t+k}}\right] \geq 0, \quad \forall t, \quad \text{where} \quad \beta^k \Lambda_{t,t+k} \equiv \frac{\Lambda_{t+k}P_{t+k}}{\Lambda_t P_t} \equiv \beta^k \frac{U_{c,t+k}}{U_{c,t}}$$

$\Lambda_{t,t+k}$ is the Marginal Rate of Intertemporal Substitution (MRIS) between time t and $t+k$ =(real) price at t of one good unit delivered at $t+k$. Λ_t and Λ_{t+k} are the Lagrange multipliers associated with the budget constraints at t and $t+k$, respectively. The No-Ponzi-Game condition ensures that the household's debt does not explode over time.

2.1.1 Intratemporal Household Problem: Allocation of expenditures

The household decision problem can be dealt with in two stages. First, we start with the optimal allocation of a given consumption expenditure across the individual goods in the consumption basket. At each period t, the household chooses varieties $C_t(i)$ such that it minimizes the expenditure needed to obtain a given level of the aggregate consumption C_t^*, i.e. taking as given $P_t(i)$.[3]

$$\min_{C_t(i)} \int_0^1 P_t(i)C_t(i)di \tag{3}$$

s.t.

$$\left(\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}} \equiv C_t \geq C_t^*$$

The Lagrangian of this problem writes

$$\mathcal{L} = \int_0^1 P_t(i)C_t(i)di + \lambda_t \left(C_t^* - \left(\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}\right)$$

with associated First Order Condition (FOC)

$$\begin{aligned} P_t(i) &= \lambda_t C_t(i)^{-\frac{1}{\epsilon}}\left(\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{1}{\epsilon-1}} \\ &= \lambda_t C_t(i)^{-\frac{1}{\epsilon}} C_t^{\frac{1}{\epsilon}} \\ &\Rightarrow \quad C_t(i) = \left(\frac{P_t(i)}{\lambda_t}\right)^{-\epsilon} C_t, \quad \forall i,t \end{aligned} \tag{4}$$

[3]There are actually thwo equivalent ways to solve this: (i) minimizing total consumption expenditure on the variety of goods for a given level of aggregate consumption (shown below), or (ii) maximizing the utility of the representative household for a given level of aggregate consumption. This is the duality principle from Mas-Colell, Whinston and Green (1995)

Taking the ratio for two goods i and j yields

$$C_t(i) = \left(\frac{P_t(i)}{P_t(j)}\right)^{-\epsilon} C_t(j).$$

Substituting into the consumption aggregator:

$$\begin{aligned} C_t &= \left(\int_0^1 \left(\left(\frac{P_t(i)}{P_t(j)}\right)^{-\epsilon} C_t(j)\right)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} \\ &= C_t(j)P_t(j)^{\epsilon}\left(\int_0^1 P_t(i)^{1-\epsilon} di\right)^{\frac{\epsilon}{\epsilon-1}} \end{aligned}$$

Using

$$P_t = \left(\int_0^1 P_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}} \tag{5}$$

we obtain the demand function on slide 5.

$$C_t = C_t(j)P_t(j)^{\epsilon}P_t^{-\epsilon}$$

$$\Rightarrow \quad C_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} C_t, \quad \forall j \tag{6}$$

By comparing (6) with (4) we can give an interpretation to λ in this setup. λ captures the reduction in expenditure when lowering the required C_t that we want to achieve by one marginal unit. Therefore we can also interpret it as the marginal cost, or price P_t of C_t. From (6) and (5) it is straightforward to verify that

$$\begin{aligned} P_t &= \left(\int_0^1 P_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}} \\ P_t^{1-\epsilon} &= \left(\int_0^1 P_t(i)^{1-\epsilon} di\right) \\ P_t^{1-\epsilon} &= \left(\int_0^1 P_t(i)\frac{C_t(i)}{C_t P_t^{\epsilon}} di\right) \\ \Rightarrow P_t C_t &= \int_0^1 P_t(i)C_t(i) di \end{aligned} \tag{7}$$

so that we can rewrite the Budget Constraint accordingly

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + D_t, \quad \forall t \tag{8}$$

2.1.2 Inter-Temporal Household Problem

The household solves the following optimization problem:

$$\max_{C_t, N_t, B_t} \mathcal{L} \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ U\left(C_t, N_t\right) + \Lambda_t \left[B_{t-1} + W_t N_t + D_t - P_t C_t - Q_t B_t \right] \right\}$$

Where Λ_t is the Lagrangian multiplier. We obtain the FOCs:

$$\frac{\partial \mathcal{L}(\cdot)}{\partial C_t} = 0 \Longleftrightarrow \beta^t \left[U_c\left(C_t, N_t\right) - \Lambda_t P_t\right] = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial N_t} = 0 \Longleftrightarrow \beta^t \left[U_n\left(C_t, N_t\right) + \Lambda_t W_t\right] = 0 \tag{10}$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial B_t} = 0 \Longleftrightarrow -\beta^t \Lambda_t Q_t + \beta^{t+1} \Lambda_{t+1} = 0 \tag{11}$$

Optimality Conditions

1. Labour Supply condition: from (9) and (10)

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t}$$

2. Euler condition: from (9) and (11) with $Q_t = \frac{1}{1+i_t}$

$$U_{c,t} = \beta \mathbb{E}_t \left[U_{c,t+1} \frac{\left(1+i_t\right) P_t}{P_{t+1}} \right]$$

3. Transversality condition:

$$\lim_{k \to \infty} \mathbb{E}_t \left[\Lambda_{t,t+k} \frac{B_{t+k}}{P_{t+k}} \right] \leq 0$$

2.1.3 Linearized Optimality Conditions

We start by using the following functional form for the utility function:

$$U\left(C_t, N_t; Z_t\right) = \begin{cases} \left(\frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{for } \sigma \neq 1 \\ \left(\ln C_t - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{for } \sigma = 1 \end{cases}$$

where Z_t is a preference shock and $Z_t = Z_{t-1}^{\rho} e^{\varepsilon_t}$.

The optimality conditions are then given by the following system of equations:

1. Labour Supply Condition:

$$\frac{N_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t}$$

2. Euler Equation Condition:

$$Z_t C_t^{-\sigma} = \beta \mathbb{E}_t \left[Z_{t+1} C_{t+1}^{-\sigma} \frac{(1+i_t)\, P_t}{P_{t+1}} \right]$$

The linearized version of the optimality conditions is given by:

1. Linearized Labour Supply Condition:

$$\sigma c_t + \varphi n_t = w_t - p_t$$

2. Linearized Euler Equation Condition: (See Appendix for linearization methods)

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - \rho\right) + \frac{1}{\sigma}\left(1-\rho_z\right) z_t$$

where

$$y_t = c_t, \quad \pi_t \equiv \ln \Pi_t = p_t - p_{t-1} \quad \text{and} \quad \rho \equiv -\ln \beta$$

2.1.4 Natural Equilibrium

Natural rate r_t^n adjusts to clear goods market so that

$$y_t^n = \mathbb{E}_t y_{t+1}^n - \frac{1}{\sigma}\left(r_t^n - \rho\right) + \frac{1}{\sigma}\left(1-\rho_z\right) z_t$$

2.1.5 Dynamic IS Curve

Use both Euler equations (actual - natural equilibrium) to get

$$\widetilde{y}_t = \mathbb{E}_t \widetilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n\right)$$

- $\widetilde{y}_t \equiv y_t - y_t^n$ is the output gap (a measure of misalignment between AS and AD)
- $i_t - \mathbb{E}_t \pi_{t+1} - r_t^n$ is the interest-rate gap (a measure of monetary policy tightness)

2.2 Firms

Firms operate under monopolistic competition with market power determined by ϵ and produce differentiated goods by using labor N_t as their only source of input. Technology A_t is equal among firms, and the production function takes the following form:

$$Y_t(i) = A_t N_t(i)$$

where $Y_t(i)$ stands for the production of output i and

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a, \quad a \equiv \log A$$

2.2.1 Aggregate price dynamics

Recall that

$$P_t \equiv \left(\int_0^1 P_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}$$

Sticky Prices à la Calvo 1983

- $\theta \in [0,1] =$ "stickiness"
- With probability θ, a firm must stick with its old price, i.e. $P_t(i) = P_{t-1}(i)$
- With probability $(1-\theta)$, a firm can optimize its price, P_t^*
- Demand function for each variety $i \in [0,1]$[4]: $Y_t(i) = C_t(i) = C_t P_t^\varepsilon P_t(i)^{-\varepsilon}$

Equation (12) displays the aggregate price index under the Calvo pricing assumption.

$$P_t = \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta) P_t^{*1-\epsilon}\right)^{\frac{1}{1-\epsilon}} \tag{12}$$

Where P_t^* is the optimal price chosen by the optimizing firms. As $\theta \to 0, P_t = P_t^*$ implies that all the firms can reset their prices as in a flexible price economy. By dividing both sides by P_{t-1}, equation (12) can also be rewritten in terms of gross inflation,$\Pi_t = \frac{P_t}{P_{t-1}}$.

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon} \tag{13}$$

[4]See Equation (6)

We log-linearize condition (12) around steady-state with zero inflation ($\bar{\Pi} = 1$).

$$
\begin{aligned}
P_t &= \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta) P_t^{*1-\epsilon}\right)^{\frac{1}{1-\epsilon}} \\
P_t^{1-\varepsilon} &= \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta) P_t^{*1-\epsilon}\right) \\
1 &= \theta \left(\frac{P_{t-1}}{P_t}\right)^{1-\varepsilon} + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\varepsilon} \\
1 &= \theta e^{(1-\varepsilon)(p_{t-1}-p_t)} + (1-\theta) e^{(1-\varepsilon)(p^*-p_t)}
\end{aligned}
$$

Next, we take the Taylor expansion and obtain:

$$
\begin{aligned}
1 &= \theta + (1-\theta) + \theta e^0 (1-\varepsilon)\left(p_{t-1} - \bar{p}\right) - \theta e^0 (1-\varepsilon)\left(p_t - \bar{p}\right) \\
&\quad + (1-\theta) e^0 (1-\varepsilon)\left(p_t^* - \bar{p}\right) - (1-\theta) e^0 (1-\varepsilon)\left(p_t - \bar{p}\right) \\
1 &= 1 + (1-\varepsilon)\Bigg(\theta\Big(\left(p_{t-1} - \bar{p}\right) - \left(p_t - \bar{p}\right)\Big) + (1-\theta)\left(p_t^* - \bar{p}\right) - \left(p_t - \bar{p}\right)\Bigg) \\
0 &= \theta(p_{t-1} - p_t) + (1-\theta)(p_t^* - p_t) \\
p_t &= \theta p_{t-1} + (1-\theta) p_t^* \\
\pi_t &= (1-\theta)\left(p_t^* - p_{t-1}\right) \qquad (14)
\end{aligned}
$$

2.2.2 Optimal Price Setting

Firm i 's value:

$$
V_t(i) = \mathbb{E}_t \sum_{k=0}^{\infty} \Lambda_{t,t+k} \left(\frac{P_{t+k}(i) - W_{t+k}/A_{t+k}}{P_{t+k}}\right) Y_{t+k}(i)
$$

- W_t/A_t = nominal marginal cost (divided by P_t gives real marginal cost)
- $\mathcal{M}_t(i) \equiv \frac{P_t(i)}{W_t/A_t}$ = firm i 's markup (factor) $\Rightarrow P_t(i) - W_t/A_t = \left(\mathcal{M}_t(i) - 1\right) \frac{W_t}{A_t}$

A price-resetting firm solves

$$
\max_{P_t^*} \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} \left(\frac{P_t^* - W_{t+k}/A_{t+k}}{P_{t+k}}\right) Y_{t+k|t} \qquad (15)
$$

Where $\Lambda_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}}$ is the stochastic discount factor. We assume that households own firms, so we assume that the two agents have the same discount factor.

$$
Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k}, \quad C_{t+k} \text{ given}
$$

Substitute, rearrange:

$$\max_{P_t^*} \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} \left(P_t^{*1-\epsilon} - \frac{W_{t+k}}{A_{t+k}} P_t^{*-\epsilon} \right) \frac{C_{t+k}}{P_{t+k}^{1-\epsilon}}$$

First Order Condition:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} \left((1-\epsilon) P_t^{*-\epsilon} + \epsilon \frac{W_{t+k}}{A_{t+k}} P_t^{*-\epsilon-1} \right) \frac{C_{t+k}}{P_{t+k}^{1-\epsilon}} = 0$$

or

$$\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} Y_{t+k|t} \frac{P_t^* - \frac{\epsilon}{\epsilon-1} W_{t+k}/A_{t+k}}{P_{t+k}} = 0 \tag{16}$$

Flexible prices $(\theta = 0)$:

$$P_t^* = \underbrace{\frac{\epsilon}{\epsilon - 1}}_{\text{desired markup } \mathcal{M}} \times \frac{W_t}{A_t}$$

Perfect competition $(\epsilon \to \infty)$ **:**

$$P_t^* \to \frac{W_t}{A_t}$$

Monopoly power implies $P_t^* > W_t/A_t$, while Calvo pricing implies that the FOC equates to zero a weighted sum of excess markups (the $P_t^* - \frac{\epsilon}{\epsilon-1} W_{t+k}/A_{t+k}$).

2.2.3 Linearize Pricing Rule

We take the optimality condition (16) associated with the firm problem and split the two components this way:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left(\Lambda_{t,t+k} Y_{t+k|t} \frac{P_t^*}{P_{t+k}} \right) = \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left(\Lambda_{t,t+k} Y_{t+k|t} \frac{\frac{\epsilon}{\epsilon-1} W_{t+k}/A_{t+k}}{P_{t+k}} \right) \tag{17}$$

Steady state: $P_t = P_t^* = P_{t+k} = \bar{P}$, $Y_{t+k|t} = Y$, $W_{t+k|t} = \bar{W}$, $A_{t+k|t} = \bar{A}$, $\Lambda_{t,t+k} = \beta^k$.

Taylor expansion on both sides of (17):

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left(\Lambda_{t,t+k}Y_{t+k|t}\frac{P_t^*}{P_{t+k}}\right) = &\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\beta^k\bar{Y}\frac{\bar{P}}{\bar{P}}\\ &+\theta^k\mathbb{E}_t\bar{Y}\frac{\bar{P}}{\bar{P}}\left(\Lambda_{t,t+k}-\beta^k\right)\\ &+\theta^k\mathbb{E}_t\beta^k\frac{\bar{P}}{\bar{P}}\left(Y_{t+k|t}-\bar{Y}\right)\\ &+\theta^k\mathbb{E}_t\beta^k\bar{Y}\frac{1}{\bar{P}}\left(P_t^*-\bar{P}\right)\\ &-\theta^k\mathbb{E}_t\beta^k\bar{Y}\frac{\bar{P}}{\bar{P}^2}\left(P_{t+k}-\bar{P}\right)\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left(\Lambda_{t,t+k}Y_{t+k|t}\frac{\frac{\epsilon}{\epsilon-1}W_{t+k}/A_{t+k}}{P_{t+k}}\right) = &\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\beta^k\bar{Y}\frac{\frac{\epsilon}{\epsilon-1}\bar{W}/\bar{A}}{\bar{P}}\\ &+\theta^k\mathbb{E}_t\bar{Y}\frac{\frac{\epsilon}{\epsilon-1}\bar{W}/\bar{A}}{\bar{P}}\left(\Lambda_{t,t+k}-\beta^k\right)+\\ &+\theta^k\mathbb{E}_t\beta^k\frac{\frac{\epsilon}{\epsilon-1}\bar{W}/\bar{A}}{\bar{P}}\left(Y_{t+k|t}-\bar{Y}\right)\\ &+\theta^k\mathbb{E}_t\beta^k\bar{Y}\frac{\frac{\epsilon}{\epsilon-1}\frac{1}{\bar{A}}}{\bar{P}}\left(W_{t+k}-\bar{W}\right)\\ &-\theta^k\mathbb{E}_t\beta^k\bar{Y}\frac{\frac{\epsilon}{\epsilon-1}\frac{\bar{W}}{\bar{A}^2}}{\bar{P}}\left(A_{t+k}-\bar{A}\right)\\ &-\theta^k\mathbb{E}_t\beta^k\bar{Y}\frac{\frac{\epsilon}{\epsilon-1}\bar{W}/\bar{A}}{\bar{P}^2}\left(P_{t+k}-\bar{P}\right)\end{aligned}$$

By elimination and $\bar{P}=\frac{\epsilon}{\epsilon-1}\frac{\bar{W}}{\bar{A}}$, the Taylor expansion of (17) delivers:

$$\begin{aligned}\sum_{k=0}^{\infty}\mathbb{E}_t\theta^k\beta^k\bar{Y}\frac{1}{\bar{P}}\left(P_t^*-\bar{P}\right) &= \sum_{k=0}^{\infty}\mathbb{E}_t\theta^k\beta^k\bar{Y}\frac{\frac{\epsilon}{\epsilon-1}\bar{W}/\bar{A}}{\bar{P}}\left(\frac{\left(W_{t+k}-\bar{W}\right)}{W}-\frac{\left(A_{t+k}-\bar{A}\right)}{A}\right)\\ \sum_{k=0}^{\infty}\mathbb{E}_t\theta^k\beta^k\hat{p}_t^* &= \sum_{k=0}^{\infty}\mathbb{E}_t\theta^k\beta^k\left(\hat{w}_{t+k}-\hat{a}_{t+k}\right)\\ \hat{p}_t^*\sum_{k=0}^{\infty}\beta^k\theta^k &= \sum_{k=0}^{\infty}\mathbb{E}_t\beta^k\theta^k\left(\hat{w}_{t+k}-\hat{a}_{t+k}\right)\\ \Longleftrightarrow p_t^* &= (1-\beta\theta)\sum_{k=0}^{\infty}\beta^k\theta^k\mathbb{E}_t\left(w_{t+k}-a_{t+k}\right)\\ \Longleftrightarrow p &= (1-\beta\theta)\sum_{k=0}^{\infty}\beta^k\theta^k\mathbb{E}_t\left(w-a\right)\end{aligned}$$

Combining these last two equations, we obtain the pricing rule:

$$p_t^* - p = (1+\beta\theta)\sum_{k=0}^{\infty}\beta^k\theta^k\mathbb{E}_t\left((w_{t+k}-a_{t+k})-(w-a)\right)$$

with $\mu = p - w - a$

$$p_t^* = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}\beta^k\theta^k\mathbb{E}_t\left(w_{t+k}-a_{t+k}\right) \tag{18}$$

Rearrange,

$$\begin{aligned}
p_t^* &= (1-\beta\theta)\mathbb{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\left(\mu + w_{t+k} - a_{t+k}\right)\\
p_t^* &= \beta\theta p_{t+1}^* + (1-\beta\theta)(\mu + \underbrace{w_t - a_t)}_{\text{log nominal marginal cost}}\\
&= \beta\theta p_{t+1}^* + (1-\beta\theta)(\mu + p_t + \underbrace{w_t - p_t - a_t}_{\text{log real marginal cost}})\\
&= \beta\theta p_{t+1}^* + (1-\beta\theta)(\mu + p_t - \underbrace{(a_t + p_t - w_t)}_{\text{log average markup } \mu_t})\\
&= \beta\theta p_{t+1}^* + (1-\beta\theta)\left(p_t - \widehat{\mu}_t\right), \quad \widehat{\mu}_t \equiv \mu_t - \mu
\end{aligned} \tag{19}$$

2.2.4 New Keynesian Philips Curve

- pricing rule: $p_t^* = \beta\theta p_{t+1}^* + (1-\beta\theta)\left(p_t - \widehat{\mu}_t\right)$
- price level: $p_t = \theta p_{t-1} + (1-\theta)p_t^*$

Solve for $\pi_t \equiv p_t - p_{t-1}$ as a function of $\widehat{\mu}_t$:

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} - \frac{(1-\theta)(1-\beta\theta)}{\theta}\widehat{\mu}_t$$

Average markup:

$$\begin{aligned}
\mu_t &= a_t - (w_t - p_t)\\
&= a_t - (\sigma c_t + \varphi n_t) \qquad \text{(labor supply)}\\
&= a_t - \sigma y_t - \varphi\left(y_t - a_t\right) \qquad \text{(labor demand)}\\
&= (1+\varphi)a_t - (\sigma+\varphi)y_t
\end{aligned}$$

Markup with natural output (= (hypothetical) output under flexible prices). All firms set p_t^* so that $\mu_t = \mu$:

$$\mu = (1+\varphi)a_t - (\sigma+\varphi)y_t^n$$

Solving for y_t^n

$$y_t^n = \underbrace{-\left(\frac{\mu}{\sigma+\varphi}\right)}_{\text{decreasing in } \mu} + \underbrace{\left(\frac{1+\varphi}{\sigma+\varphi}\right)}_{\equiv \psi_{ya} > 0} a_t$$

We infer the natural interest rate:

$$\begin{aligned} r_t^n &= \rho + (1-\rho_z)\, z_t + \sigma \mathbb{E}_t \Delta y_{t+1}^n \\ &= \rho + (1-\rho_z)\, z_t + \sigma \psi_{ya} (\rho_a - 1)\, a_t \end{aligned}$$

From the expressions for μ_t and μ we get

$$\underbrace{\mu_t - \mu}_{\widehat{\mu}_t} = -(\sigma+\varphi) \underbrace{(y_t - y_t^n)}_{\text{output gap } \widetilde{y}_t}$$

and therefore we obtain the New Keynesian Philips Curve

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \widetilde{y}_t$$

where

$$\kappa \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}(\sigma+\varphi) \geq 0$$

2.3 Monetary policy rule

Model closed by interest-rate rule $i_t = f(\Omega_t, v_t)$ where

- Ω_t = set of observables (e.g., π_t)
- v_t = monetary policy shock (i.e., residual of policy rule)

Assume, for example:

$$i_t = \rho + \phi_\pi \pi_t + \phi_y (y_t - y) + v_t$$

where

$$v_t = \rho_v v_{t-1} + \varepsilon_t^v$$

Rewrite $y_t - y$ as follows:

$$y_t - y = \underbrace{y_t - y_t^n}_{=\widetilde{y}_t} + \underbrace{y_t^n - y}_{=\widehat{y}_t^n=\psi_{ya}a_t} = \widetilde{y}_t + \psi_{ya}a_t$$

$$\Rightarrow i_t = \rho + \phi_\pi \pi_t + \phi_y \left(\widetilde{y}_t + \psi_{ya}a_t\right) + v_t$$

2.4 Model Summary

- Dynamic IS curve:

$$\widetilde{y}_t = \mathbb{E}_t \widetilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n\right) \tag{20}$$

- Natural rate of interest:

$$r_t^n = \rho - \sigma\left(1 - \rho_a\right)\psi_{ya}a_t + \left(1 - \rho_z\right)z_t$$

- NKPC:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \widetilde{y}_t \tag{21}$$

- Monetary policy rule:

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \widetilde{y}_t + \phi_y \psi_{ya} a_t + v_t \tag{22}$$

- Shocks:

$$z_t = \rho_z z_{t-1} + \varepsilon_t^z$$
$$a_t = \rho_a a_{t-1} + \varepsilon_t^a$$
$$v_t = \rho_v v_{t-1} + \varepsilon_t^v$$

2.4.1 Matrix Form

It is convenient to work with a reduced form representation of (20) and (21) who take into account the policy rule (22) under consideration. Plugging (22) in (20), we obtain the following equation for $\tilde{y}_t$:

$$
\begin{aligned}
\tilde{y}_t &= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - r_t^n\right) \\
&= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(\rho + \phi_\pi\pi_t + \phi_y\tilde{y}_t + \phi_y\psi_{ya}a_t + v_t - E_t\pi_{t+1} - \left(\rho - \sigma\left(1-\rho_a\right)\psi_{ya}a_t + \left(1-\rho_z\right)z_t\right)\right) \\
&= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(\phi_\pi\pi_t - E_t\pi_{t+1} + \phi_y\tilde{y}_t + (\phi_y + \sigma\left(1-\rho_a\right))\psi_{ya}a_t + \left(1-\rho_z\right)z_t + v_t\right) \\
&= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(\phi_\pi(\beta E_t\pi_{t+1} + \kappa\tilde{y}_t) - E_t\pi_{t+1} + \phi_y\tilde{y}_t + (\phi_y + \sigma\left(1-\rho_a\right))\psi_{ya}a_t + \left(1-\rho_z\right)z_t + v_t\right) \\
&= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left((\phi_\pi\beta - 1)E_t\pi_{t+1} + (\phi_y + \phi_\pi\kappa)\tilde{y}_t + (\phi_y + \sigma\left(1-\rho_a\right))\psi_{ya}a_t + \left(1-\rho_z\right)z_t + v_t\right) \\
\tilde{y}_t &= E_t\tilde{y}_{t+1} + \frac{1-\beta\phi_\pi}{\sigma}E_t\pi_{t+1} - \frac{\phi_\pi\kappa + \phi_y}{\sigma}\tilde{y}_t + \frac{(\phi_y + \sigma\left(1-\rho_a\right)\psi_{ya}a_t + \left(1-\rho_z\right)z_t + v_t)}{\sigma}
\end{aligned}
$$

$$
\frac{\sigma + \phi_\pi\kappa + \phi_y}{\sigma}\tilde{y}_t = E_t\tilde{y}_{t+1} + \frac{1-\beta\phi_\pi}{\sigma}E_t\pi_{t+1} + \frac{(\phi_y + \sigma\left(1-\rho_a\right)\psi_{ya}a_t + \left(1-\rho_z\right)z_t + v_t)}{\sigma}
$$

$$
\tilde{y}_t = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}\left[\sigma E_t\tilde{y}_{t+1} + \left(1-\beta\phi_\pi\right)E_t\pi_{t+1} + u_t\right] \tag{23}
$$

where u_t is a composite shock term: $(\phi_y + \sigma\left(1-\rho_a\right)\psi_{ya}a_t + \left(1-\rho_z\right)z_t + v_t)$.
Equation (23) shows the current output gap as a function of the expected output gap, expected inflation, and shocks. We next achieve a similar representation of current inflation. Insert (23) into (21) and get:

$$
\begin{aligned}
\pi_t &= \beta E_t\pi_{t+1} + \kappa\left\{\frac{1}{\sigma + \phi_y + \kappa\phi_\pi}\left[\sigma E_t\tilde{y}_{t+1} + \left(1-\beta\phi_\pi\right)E_t\pi_{t+1} + u_t\right]\right\} \\
&= \frac{\sigma\kappa}{\sigma + \phi_y + \kappa\phi_\pi}E_t\tilde{y}_{t+1} + \frac{\kappa\left(1-\beta\phi_\pi\right) + \beta\left(\sigma + \phi_y + \kappa\phi_\pi\right)}{\sigma + \phi_y + \kappa\phi_\pi}E_t\pi_{t+1} + \frac{\kappa}{\sigma + \phi_y + \kappa\phi_\pi}u_t \\
&= \frac{\sigma\kappa}{\sigma + \phi_y + \kappa\phi_\pi}E_t\tilde{y}_{t+1} + \frac{\kappa + \beta\left(\sigma + \phi_y\right)}{\sigma + \phi_y + \kappa\phi_\pi}E_t\pi_{t+1} + \frac{\kappa}{\sigma + \phi_y + \kappa\phi_\pi}u_t \\
\pi_t &= \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}\left(\sigma\kappa E_t\tilde{y}_{t+1} + \left[\kappa + \beta\left(\sigma + \phi_y\right)\right]E_t\pi_{t+1} + \kappa u_t\right) \qquad (24)
\end{aligned}
$$

Finally, the two equations (23) and (24) can be written as a system of forward looking difference equations:

$$\begin{aligned}
\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} &= \frac{1}{\sigma+\phi_y+\phi_\pi\kappa}\begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \sigma\kappa & \kappa+\beta\left(\sigma+\phi_y\right) \end{bmatrix}\begin{bmatrix} E_t\tilde{y}_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} + \frac{1}{\sigma+\phi_y+\phi_\pi\kappa}\begin{bmatrix} 1 \\ \kappa \end{bmatrix} u_t \\
&= \Omega\begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \sigma\kappa & \kappa+\beta\left(\sigma+\phi_y\right) \end{bmatrix}\begin{bmatrix} E_t\tilde{y}_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} + \Omega\begin{bmatrix} 1 \\ \kappa \end{bmatrix} u_t \\
&= \boldsymbol{A}_T\begin{bmatrix} E_t\tilde{y}_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} + \boldsymbol{B}_T u_t \qquad (25)
\end{aligned}$$

The system is a reduced form representation of the dynamic IS curve and the New Keynesian Phillips curve, which takes into account effects from the policy defined in equation (22). The coefficient matrix $\boldsymbol{A}_T$ represents effects from expectations on current output gap and inflation while the coefficient vector $\boldsymbol{B}_T$ represents the effects from shocks in $u_t = (\phi_y + \sigma\left(1-\rho_a\right)\psi_{ya}a_t + \left(1-\rho_z\right)z_t + v_t)$.

2.4.2 Method of Undetermined Coefficients: for monetary policy shock

Assume that the exogenous component of (22) follows an AR(1) process, where $\rho_v \in [0,1)$:

$$v_t = \rho_v v_{t-1} + \varepsilon_t^v$$

Notice that a positive (negative) realization of ε_t^v is interpreted as a contractionary (expansionary) monetary policy shock, leading to a rise (decline) in the nominal interest rate for given levels of inflation and output gap. We want to find the contemporaneous effects of a monetary policy shock v_t on the output gap $\tilde{y}_t$ and inflation π_t. One way to identify these effects is by using the method of undetermined coefficients.

1. First, we guess:

$$\tilde{y}_t = \psi_{yv}\rho_v v_t, \quad \tilde{\pi}_t = \psi_{\pi v}v_t, \quad E_t\tilde{y}_{t+1} = \psi_{yv}\rho_v v_t, \quad E_t\tilde{\pi}_t = \rho_v\psi_{\pi v}v_t \qquad (26)$$

2. In the second step, we substitute the above relations into the equations:

$$\psi_{yv}v_t = \psi_{yv}\rho_v v_t - \frac{1}{\sigma}\left(\phi_\pi \psi_{\pi v} v_t + \phi_y \psi_{yv} v_t - \psi_{\pi v}\rho_v v_t + v_t\right) \tag{27}$$

$$\psi_{\pi v}v_t = \beta\psi_{\pi v}\rho_v v_t + k\psi_{yv}v_t \tag{28}$$

3. From (28) we get:

$$\psi_{yv} = \frac{1-\beta\rho_v}{\kappa}\psi_{\pi v}$$

4. Finally, in step four, we insert (26) in (23) solve for the coefficient

$$\begin{aligned}
\psi_{yv}v_t &= E_t\psi_{yv}v_{t+1} - \frac{1}{\sigma}\left(\phi_\pi\psi_{\pi v}v_t + \phi_y\psi_{yv}v_t + v_t - E_t\psi_{\pi v}v_{t+1}\right)\\
&= \psi_{yv}\rho_v v_t - \frac{1}{\sigma}\left(\phi_\pi\psi_{\pi v}v_t + \phi_y\psi_{yv}v_t + v_t - \psi_{\pi v}\rho_v v_t\right)\\
&= \frac{\sigma\rho_v - \phi_y}{\sigma}\psi_{yv}v_t - \frac{\phi_\pi - \rho_v}{\sigma}\psi_{\pi v}v_t - \frac{1}{\sigma}v_t\sigma\psi_{yv}\\
-1 &= \left(\sigma\rho_v - \phi_y\right)\psi_{yv} - \left(\phi_\pi - \rho_v\right)\psi_{\pi v}\\
-1 &= \left[\sigma\left(1-\rho_v\right) + \phi_y\right]\psi_{yv} + \left(\phi_\pi - \rho_v\right)\psi_{\pi v}\\
-1 &= \left[\sigma\left(1-\rho_v\right) + \phi_y\right]\frac{1-\beta\rho_v}{\kappa}\psi_{\pi v} + \left(\phi_\pi - \rho_v\right)\psi_{\pi v}\\
-1 &= \frac{\left(1-\beta\rho_v\right)\left[\sigma\left(1-\rho_v\right)+\phi_y\right] + \kappa\left(\phi_\pi - \rho_v\right)}{\kappa}\psi_{\pi v}\\
\Rightarrow \psi_{\pi v} &= \frac{-\kappa}{\left(1-\beta\rho_v\right)\left[\sigma\left(1-\rho_v\right)+\phi_y\right] + \kappa\left(\phi_\pi - \rho_v\right)} = -\kappa\Lambda_v
\end{aligned}$$

So we can write:

$$\psi_{yv} = -\Lambda_v(1-\beta\rho_v), \quad \psi_{\pi v} = -k\Lambda_v$$

Finally, this means that the solutions are:

$$\begin{aligned}
\tilde{y}_t &= -\left(1-\beta\rho_v\right)\Lambda_v v_t\\
\pi_t &= -\kappa\Lambda_v v_t
\end{aligned}$$

To ease the notation, $\Lambda_v \equiv \frac{1}{(1-\beta\rho_v)[\sigma(1-\rho_v)+\phi_y]+\kappa(\phi_\pi-\rho_v)}$.

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A Linearization Methods

Linearization (and log-linearization) essentially consists of doing a Taylor expansion of a non-linear function. Suppose $f : \mathbb{R}^n \to \mathbb{R}$, then

$$f(x_t^1, ..., x_t^n) \approx f(x^1, ..., x^n) + \sum_{i=1}^{n} \frac{\partial}{\partial x^i} f(x^1, ..., x^n)(x_t^i - x^i)\,, \tag{1}$$

is a Taylor expansion of order 1 of the function f around the point $(x^1, ..., x^n)$.

Consider a level variable denoted X_t. Define $x_t = \log(X_t)$ and note X $(x = \log(X))$ its steady-state value. We define the hat variable, $\hat{x}_t$, as the log-deviation of X_t from its steady-state value, that is

$$\hat{x}_t = \log(X_t) - \log(X) = x_t - x\,. \tag{2}$$

From (2), we can derive two important formulas that will prove very helpful. Note that (2) is equivalent to

$$\hat{x}_t = \log\left(\frac{X_t}{X}\right) = \log\left(1 + \frac{X_t - X}{X}\right)\,,$$

and

$$\log(1 + x) \approx \log(1) + \frac{1}{1+0}(x - 0) = x\,.$$

We then obtain that

$$\hat{x}_t \approx \frac{X_t - X}{X}\,. \tag{3}$$

From (2) we can also get the following formula

$$X_t = Xe^{\hat{x}_t} \approx X(e^0 + e^0(\hat{x}_t - 0)) = X(1 + \hat{x}_t)\,, \tag{4}$$

where the Taylor approximation was done around 0 as we have just shown above that the steady state of a hat variable is always zero. The last equality provides a direct formula to log-linearize without having to perform the Taylor expansion yourself. I would always recommend doing these things yourself, but you may find some lecture notes that directly apply the formula $X_t = X(1 + \hat{x}_t)$. The bottom line is that you should try different methods and understand that if done correctly they all yield the same result. Below I show how to log-linearize the Euler equation using multiple methods.

A.1 Example: Consumption Euler equation

Let us consider the standard Euler equation with $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ and we do not have a preference shock Z_t as on page 7 of the lecture notes.

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} \frac{(1+i_t)}{\Pi_{t+1}} \right]$$

where $\sigma > 0$ is the coefficient of relative risk aversion. We can rewrite the Euler equation as:

$$1 = \beta \mathbb{E}_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \frac{(1+i_t)}{\Pi_{t+1}} \right] \tag{5}$$

In steady-state, the Euler equation becomes:

$$1 = \beta \left[\left(\frac{C^*}{C^*} \right)^{\sigma} \frac{(1+i^*)}{\Pi^*} \right]$$

where C^* is the steady-state value of consumption, i^* is the steady-state value of the nominal interest rate and $\Pi^* = 1$ is the steady-state value of inflation. We find that in steady-state:

$$\begin{aligned} 1 &= \beta(1+i^*) \\ \frac{1}{1+\rho} &= \beta \quad \Rightarrow i^* = \rho \end{aligned}$$

A.1.1 Ordinary Taylor exapansion

First, we can do a first-order Taylor expansion of the Euler equation (5) around the steady-state:

$$\begin{aligned} 1 = \beta \left[\left(\frac{C^*}{C^*} \right)^{\sigma} \frac{(1+i^*)}{\Pi^*} \right] &+ \beta \left(\frac{\sigma(1+i^*)}{\Pi^*} \left(\frac{C^*}{C^*} \right)^{\sigma-1} ((C_t - C^*) - (\mathbb{E}_t C_{t+1} - C^*)) \right) \\ &+ \beta \left[\left(\frac{C^*}{C^*} \right)^{\sigma} \frac{1}{\Pi^*} \right] (i_t - i^*) \\ &- \beta \left(\left(\frac{C^*}{C^*} \right)^{\sigma} \frac{(1+i^*)}{\Pi^*} \left(\frac{\mathbb{E}_t \Pi_{t+1} - \Pi^*}{\Pi^*} \right) \right) \end{aligned}$$

Some terms cancel, and we approximate the term $\frac{1}{1+i^*} = 1$. If the discount factor is sufficiently high, this will be a good approximation.[5] Then, we can rearrange the equation as follows:

$$
\begin{aligned}
0 &= \frac{\sigma(C_t - C^*)}{C^*} - \frac{\sigma(\mathbb{E}_t C_{t+1} - C^*)}{C^*} + (i_t - i^*) - \frac{\mathbb{E}_t \Pi_{t+1} - \Pi^*}{\Pi^*} \\
0 &= \sigma \hat{c}_t - \sigma \mathbb{E}_t \hat{c}_{t+1} + (i_t - i^*) - \mathbb{E}_t \hat{\pi}_{t+1} \\
0 &= \sigma \left(c_t - c\right) - \sigma \left(\mathbb{E}_t c_{t+1} - c\right) + (i_t - i^*) - \left(\mathbb{E}_t \pi_{t+1} - \pi\right)
\end{aligned}
$$

since $\pi = 0$ and σc terms cancel each other out, we obtain (6)

$$
\begin{aligned}
0 &= \sigma c_t - \sigma(\mathbb{E}_t c_{t+1} + (i_t - i^*) - \mathbb{E}_t \pi_{t+1} \\
c_t &= \mathbb{E}_t c_{t+1} - \frac{1}{\sigma}\left(i_t - \rho - \mathbb{E}_t \pi_{t+1}\right) \qquad (7)
\end{aligned}
$$

A.1.2 Log transformation

First, we can take logs of both sides of the Euler equation (5):

$$
\begin{aligned}
\ln 1 &= \ln \beta + \ln\left(1 + i_t\right) + \sigma \ln C_t - \sigma \mathbb{E}_t \ln C_{t+1} - \mathbb{E}_t \ln \Pi_{t+1} \\
\ln 1 &= \ln \beta - \sigma \mathbb{E}_t c_{t+1} + \sigma c_t + i_t - \mathbb{E}_t \pi_{t+1}
\end{aligned}
$$

Note that, in the steady state, $1 + i^* = \frac{1}{\beta} = 1 + \rho$, hence $i^* = \ln\left(1 + i^*\right) = -\ln \beta$. Using this, we obtain the same equation as in (6):

$$
\begin{aligned}
\sigma c_t &= \sigma \mathbb{E}_t\left(c_{t+1}\right) + i^* - i_t + \mathbb{E}_t \pi_{t+1} \\
c_t &= \mathbb{E}_t\left(c_{t+1}\right) - \frac{1}{\sigma}\left(i_t - i^* - \mathbb{E}_t \pi_{t+1}\right) \\
c_t &= \mathbb{E}_t\left(c_{t+1}\right) - \frac{1}{\sigma}\left(i_t - \rho - \mathbb{E}_t \pi_{t+1}\right)
\end{aligned}
$$

A.1.3 Log linearization

We again start from the Euler equation (5), which we can rewrite as:

$$
1 = \beta e^{\left(\ln(1+i_t) + \sigma \ln C_t - \sigma \mathbb{E}_t \ln C_{t+1} - \mathbb{E}_t \ln \Pi_{t+1}\right)} = \beta e^{\left(i_t + \sigma c_t - \sigma \mathbb{E}_t c_{t+1} - \mathbb{E}_t \pi_{t+1}\right)} \qquad (8)
$$

[5]See notes by Chris Sims, https://sites.nd.edu/esims/files/2023/05/log$_l$$inearization_s$$p17.pdf$

Now, we can do a first-order Taylor approximation of (7) around the steady state:

$$
\begin{aligned}
1 &= \beta e^{(i^*+\sigma c^*-\sigma c^*-\pi^*)} \\
&\quad + \beta e^{(i^*+\sigma c^*-\sigma c^*-\pi^*)}\Big(\sigma(c_t - c^*) - \sigma(\mathbb{E}_t c_{t+1} - c^*) + (i_t - i^*) - (\mathbb{E}_t \pi_{t+1} - \pi^*)\Big) \\
1 &= 1 + 1\Big(\sigma(c_t - c^*) - \sigma(\mathbb{E}_t c_{t+1} - c^*) + (i_t - i^*) - (\mathbb{E}_t \pi_{t+1} - \pi^*)\Big) \\
0 &= \sigma c_t - \sigma \mathbb{E}_t c_{t+1} + (i_t - i^*) - \mathbb{E}_t \pi_{t+1} \\
c_t &= \mathbb{E}_t c_{t+1} - \frac{1}{\sigma}(i_t - \rho - \mathbb{E}_t \pi_{t+1})
\end{aligned}
$$